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LETTER TO THE EDITOR

The embedding SO(4) pseudoparticle solutions to the Yang–Mills equations

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Abstract. In a recent paper De, Konar and Ray obtained the general embedding SO(4) solutions to the equation which we derived for the general spherically symmetric gauge potentials in the SU(N) sourceless Yang–Mills theory. In this letter we discuss their physical meaning and point out that the only analytic solutions are the instanton ones.

In previous papers (Ma and Xu 1984a, b) we proved in terms of the method of the phase factor of the standard differential loop (Gu 1981, Ma 1984) that the general spherically symmetric gauge potential must be synchrospherically symmetric, i.e. there is a gauge (central gauge) in which the SU(N) gauge potentials before and after the rotation in the four-dimensional Euclidean space are related by a global gauge transformation which is an N-dimensional representation $\mathscr{D}$ of the SO(4) group:

$$R_{\mu\nu} W_\nu (R^{-1}x) = \mathscr{D}(R^{-1}) W_\mu (x) \mathscr{D}(R). \tag{1}$$

If one makes a constraint that the potential $W_\mu(x)$ is composed of the generators $I_{\mu\nu}$ of $\mathscr{D}$, the general forms of $W_\mu(x)$ are proved to be

$$W_\mu(x) = \varphi_1(x^2) I_{\mu\nu} x_\nu + \varphi_2(x^2) \tfrac{1}{2} \varepsilon_{\mu\nu\rho\lambda} I_{\nu\rho} x_\lambda. \tag{2}$$

This constraint means that $W_\mu(x)$ in (2) is the SO(4) embedding potential. Substituting the form (2) into the sourceless Yang–Mills equation, we have (Ma and Xu 1984a, b)

$$2x^2(\ddot{\varphi}_1 \pm \ddot{\varphi}_2) + 6(\dot{\varphi}_1 \pm \dot{\varphi}_2) - e^2 x^2 (\varphi_1 \pm \varphi_2)^3 + 3e(\varphi_1 \pm \varphi_2)^2 = 0 \tag{3}$$

where dots refer to the differentiation with respect to x^2.

In the proof of (1), the consistency conditions of the gauge potential (phase factor) are used (Ma and Xu 1984a)

$$W_\mu(0) = 0 \qquad W_\mu(x) x_\mu = 0 \tag{4}$$

so $\varphi_i(0)$ is finite. If the potential $W_\mu(x)$ is singular at the origin, it is not proved that the general spherically symmetric gauge potential must be a synchrospherically symmetric one. However, the singular potential satisfying (2) is still a synchrospherically symmetric one and (3) holds for it.

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De *et al* (1986) obtained the general solutions to (3) without paying attention to the analytic property. We will show that the only analytic solutions are the instanton ones.

Let

$$x^2(\varphi_1+\varphi_2)=y_1 \qquad x^2(\varphi_1-\varphi_2)=y_2 \tag{5}$$

$$\ln x^2=t \qquad x^2=e^t \tag{6}$$

and we obtain

$$\begin{aligned} y_1' &= \mathrm{d}y_1/\mathrm{d}t = x^2[x^2(\dot\varphi_1+\dot\varphi_2)+(\varphi_1+\varphi_2)] \\ y_2' &= \mathrm{d}y_2/\mathrm{d}t = x^2[x^2(\dot\varphi_1-\dot\varphi_2)+(\varphi_1-\varphi_2)] \end{aligned} \tag{7}$$

and

$$2y_i''=e^2y_i^3-3ey_i^2+2y_i \qquad i=1,2. \tag{8}$$

Integrating it, we get

$$y_i'^2=\tfrac{1}{4}y_i^2(ey_i-2)^2+d_i \qquad i=1,2 \tag{9}$$

where d_i is the integral constant. If φ_i is finite at the origin

$$y_i=0 \qquad \text{and} \qquad y_i'=0 \qquad \text{at } x^2=0 \tag{10}$$

so

$$d_i=0 \tag{11}$$

$$y_i'=\pm\tfrac{1}{2}y_i(ey_i-2). \tag{12}$$

y_i' may change sign only at

$$y_i=0 \qquad \text{or} \qquad ey_i-2=0. \tag{13}$$

(*a*) If the minus sign in (12) is taken

$$\begin{aligned} t=\ln x^2 &= -\int\frac{2\mathrm{d}y_i}{y_i(ey_i-2)}=\ln\frac{y_i(ey_{i0}-2)}{y_{i0}(ey_i-2)} \\ y_{i0} &= y_i|_{x^2=1}. \end{aligned} \tag{14}$$

Therefore

$$y_i=\frac{2y_{i0}x^2}{ex^2y_{i0}-ey_{i0}+2} \tag{15}$$

$$\begin{aligned} \varphi_1+\varphi_2 &= \frac{2}{e(x^2+2/ey_{10}-1)} \\ \varphi_1-\varphi_2 &= \frac{2}{e(x^2+2/ey_{20}-1)}. \end{aligned} \tag{16}$$

The analytic condition of this gauge potential requires

$$\frac{2}{ey_{i0}}-1=a_i^2>0 \tag{17}$$

so these are the instanton solutions (Ma and Xu 1984a). Note that

$$y_i=\frac{2x^2}{e(x^2+a_i^2)}\geqslant 0 \qquad ey_i-2=-\frac{2a_i^2}{x^2+a_i^2}<0. \tag{18}$$

The conditions (13) are never satisfied except for the origin, i.e. the solutions (16) hold in the whole space.

(*b*) If the plus sign in (12) is taken

$$y_i = \frac{2}{e(1+x^2+2x^2/ey_{i0})} \equiv \frac{2}{e(1+a_i x^2)}$$
$$a_i \equiv \frac{2}{ey_{i0}} - 1 \tag{19}$$
$$ey_i - 2 = -\frac{2a_i x^2}{1+a_i x}$$

y_i does not vanish at the origin, so φ_i is divergent there. If $a_i < 0$, an additional divergent point $x^2 = -1/a_i$ appears.

The Yang-Mills equation is covariant either under the Lorentz transformation, or under the gauge one, or under the conformal ones. From the symmetrical consideration, the general form of the energy-momentum tensor $\theta_{\mu\nu}$ is (Barut and Xu 1982)

$$\begin{aligned}\theta_{\mu\nu} &= \tfrac{1}{2}\mathrm{Tr}[4G_{\mu\rho}G_{\nu\rho} - \delta_{\mu\nu}G_{\rho\lambda}G_{\rho\lambda}] \\ &= \theta(x^2)[4x_\mu x_\nu - x^2\delta_{\mu\nu}].\end{aligned} \tag{20}$$

For the zero-energy solution

$$E = \int \theta_{44}\, \mathrm{d}^3x = 0 \tag{21}$$

we have

$$\theta(x^2) = 0 \qquad \text{instanton-like solutions} \tag{22a}$$

or

$$\theta(x^2) \sim x^{-6} \qquad \text{meron-like solutions.} \tag{22b}$$

It is straightforward to calculate $\theta_{\mu\nu}$ from the general form (2). We have found the gauge strength $G_{\mu\nu}$ as

$$\begin{aligned}G_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu - \mathrm{i}e[W_\mu, W_\nu] \\ &= [-2\varphi_1 + ex^2(\varphi_1^2+\varphi_2^2)]I_{\mu\nu} - \varphi_2\varepsilon_{\mu\nu\rho\lambda}I_{\rho\lambda} \\ &\quad + [2\dot\varphi_1 + e(\varphi_1^2+\varphi_2^2)][x_\mu I_{\nu\rho} - x_\nu I_{\mu\rho}]x_\rho \\ &\quad + \dot\varphi_2(x_\mu\varepsilon_{\nu\sigma\rho\lambda} - x_\nu\varepsilon_{\mu\sigma\rho\lambda})I_{\sigma\rho}x_\lambda - 2e\varphi_1\varphi_2\varepsilon_{\mu\nu\sigma\lambda}x_\rho I_{\rho\sigma}x_\lambda.\end{aligned} \tag{23}$$

For the irreducible representation $\mathscr{D} = \mathscr{D}^{jk}$, we have

$$\begin{aligned}&I_{ab} = \varepsilon_{abc}(L_c + K_c) \qquad I_{a4} = L_a - K_a \\ &C_1^{jk} = \mathrm{Tr}(\boldsymbol{L}^2) = j(j+1)(2j+1)(2k+1) \\ &C_2^{jk} = \mathrm{Tr}(\boldsymbol{K}^2) = k(k+1)(2j+1)(2k+1) \\ &\mathrm{Tr}(I_{\mu\nu}I_{\rho\lambda}) = A_1(\delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\rho}) + A_2\varepsilon_{\mu\nu\rho\lambda} \\ &A_1 = \mathrm{Tr}(L_a^2) + \mathrm{Tr}(K_a^2) = \tfrac{1}{3}(C_1^{jk} + C_2^{jk}) \\ &A_2 = \mathrm{Tr}(L_a^2) - \mathrm{Tr}(K_a^2) = \tfrac{1}{3}(C_1^{jk} - C_2^{jk}) \qquad a, b = 1, 2, 3.\end{aligned} \tag{24}$$

Then

$$\begin{aligned}\mathrm{Tr}(G_{\mu\lambda}G_{\nu\lambda}) = {} & A_1 x_\mu x_\nu[8e\varphi_1(\varphi_1^2+3\varphi_2^2)-2e^2x^2(\varphi_1^4+6\varphi_1^2\varphi_2^2+\varphi_2^4)\\ & +16(\varphi_1\dot\varphi_1+\varphi_2\dot\varphi_2)+8x^2(\dot\varphi_1^2+\dot\varphi_2^2)]\\ & +A_2x_\mu x_\nu[8e\varphi_2(3\varphi_1^2+\varphi_2^2)\\ & -8e^2x^2\varphi_1\varphi_2(\varphi_1^2+\varphi_2^2)+16(\varphi_1\dot\varphi_2+\varphi_2\dot\varphi_1)+16x^2\dot\varphi_1\dot\varphi_2]\\ & +A_1\delta_{\mu\nu}[12(\varphi_1^2+\varphi_2^2)-8ex^2\varphi_1(\varphi_1^2+3\varphi_2^2)\\ & +2e^2x^4(\varphi_1^4+6\varphi_1^2\varphi_2^2+\varphi_2^4)+8x^2(\varphi_1\dot\varphi_1+\varphi_2\dot\varphi_2)+4x^4(\dot\varphi_1^2+\dot\varphi_2^2)]\\ & +A_2\delta_{\mu\nu}[24\varphi_1\varphi_2-8ex^2\varphi_2(3\varphi_1^2+\varphi_2^2)\\ & +8e^2x^4\varphi_1\varphi_2(\varphi_1^2+\varphi_2^2)+8x^2(\varphi_1\dot\varphi_2+\varphi_2\dot\varphi_1)+8x^4\dot\varphi_1\dot\varphi_2]. \end{aligned} \tag{25}$$

By making use of (7) and (9) to eliminate the derivatives $\dot\varphi_1$ and $\dot\varphi_2$, we get

$$\begin{aligned}\mathrm{Tr}(G_{\mu\lambda}G_{\nu\lambda}) = {} & \frac{4}{x^6}[A_1(d_1+d_2)+A_2(d_1-d_2)]x_\mu x_\nu\\ & +\delta_{\mu\nu}\{\tfrac{3}{2}(A_1+A_2)(\varphi_1+\varphi_2)^2[2-ex^2(\varphi_1+\varphi_2)]^2\\ & +\tfrac{3}{2}(A_1-A_2)(\varphi_1-\varphi_2)^2[2-ex^2(\varphi_1-\varphi_2)]^2\\ & +(2/x^4)[(A_1+A_2)d_1+(A_1-A_2)d_2]\}\end{aligned} \tag{26}$$

and

$$\begin{aligned}\theta_{\mu\nu} & = \frac{2}{x^6}(4x_\mu x_\nu-\delta_{\mu\nu}x^2)[A_1(d_1+d_2)+A_2(d_1-d_2)]\\ & = \frac{4}{3x^6}[d_1C_1^{jk}+d_2C_2^{jk}](4x_\mu x_\nu-\delta_{\mu\nu}x^2).\end{aligned} \tag{27}$$

Obviously, for the instanton solutions, $d_1=d_2=0$, we have $\theta(x^2)=0$. Generally, if $d_1=d_2=0$, the solutions are instanton-like ones; if $d_1C_1^{jk}+d_2C_2^{jk}\neq 0$ the solutions are meron-like ones. Most of the solutions are singular at some points.

Now we discuss the solutions divergent only at the origin. In our previous paper (Ma and Xu 1984b) we discussed the solutions with the form

$$\varphi_1\pm\varphi_2\sim 1/x^2 \tag{28}$$

and got the meron solutions

$$\varphi_1\pm\varphi_2=\frac{M_\pm}{ex^2}\qquad M_\pm=0,1,2. \tag{29}$$

For the case that φ_i is divergent at $x^2\sim 0$ ($t\sim-\infty$), the asymptotic form of y_i at $t\sim-\infty$ has to be constant because of (9) if the integral constant d_i is finite, namely

$$\begin{aligned} & y_i\sim b_i \qquad \text{at } t\sim-\infty\\ & d_i=-\tfrac{1}{4}b_i^2(eb_i-2)^2.\end{aligned} \tag{30}$$

If y_i' is equal to zero everywhere, we obtain the meron solutions (29). If y_i' is not vanishing at the large $(-t)$, we can integrate (9):

$$\int_{-\infty}^{t}\mathrm{d}t=\pm\int_{b_i}^{y}2\{[y_i(ey_i-2)+b_i(eb_i-2)](y_i-b_i)(ey_i-2+eb_i)\}^{-1/2}\,\mathrm{d}y_i. \tag{31}$$

Since the LHS of (31) is divergent at $t \sim -\infty$, b_i has to be a degenerate root of the denominator on the RHS of (31), namely

$$b_i = 1/e \qquad \text{or} \qquad 2/e. \tag{32}$$

For $b_i = 2/e$, we obtain

$$(ey_i - 2)/y_i = cx^2 \tag{33}$$

where the plus sign on the RHS of (31) is taken. Obviously, it is just the solution (19) which is instanton-like.

For $b_i = 1/e$, the right-hand side of (9) is negative when y_i leaves from $1/e$, namely (9) cannot be satisfied.

In summary, the only analytic spherically symmetric gauge potentials satisfying the SU(N) sourceless Yang–Mills equation are the instanton ones (16), and those singular only at the origin are the meron solutions (29) and instanton-like solutions (19) with $a_i > 0$.

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